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Classical R-matrix theory of dispersionless systems: I. (1 + 1)-dimension theory

Maciej Błaszak and Błażej M Szablikowski

Institute of Physics, A Mickiewicz University, Umultowska 85, 61-614 Poznań, Poland

E-mail: blaszakm@main.amu.edu.pl and bszablik@amu.edu.pl

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Abstract
A systematic way of construction of (1 + 1)-dimensional dispersionless integrable Hamiltonian systems is presented. The method is based on the classical R-matrix on Poisson algebras of formal Laurent series. Results are illustrated with the known and new (1 + 1)-dimensional dispersionless systems.

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1. Introduction

First-order PDEs of the form

$$\frac{\partial u_i}{\partial t} = \sum_{j=1}^{n} v_{ij}(u)\frac{\partial u_j}{\partial x} \qquad i = 1, \ldots, n \tag{1.1}$$

are called *hydrodynamic* or *dispersionless* systems in (1 + 1)-dimension. In this paper, we are interested in those PDEs among (1.1) which have multi-Hamiltonian structure, infinite hierarchy of symmetries and conservation laws. An important subclass of such systems is dispersionless limits of soliton equations. Differential Poisson structures for hydrodynamic systems were introduced for the first time by Dubrovin and Novikov [1] in the form

$$\pi_{ij} = g^{ij}(u)\partial_x - \sum_k \Gamma^{ij}_k(u)\frac{\partial u_k}{\partial x} \tag{1.2}$$

where g^{ij} is a contravariant flat metric and Γ^{ij}_k are related coefficients of the contravariant Levi-Civita connection. Then they were generalized by Mokhov and Ferapontov [2] to the nonlocal form

$$\pi_{ij} = g^{ij}(u)\partial_x - \sum_k \Gamma^{ij}_k(u)\frac{\partial u_k}{\partial x} + c\frac{\partial u_i}{\partial x}\partial_x^{-1}\frac{\partial u_j}{\partial x} \tag{1.3}$$

in the case of constant curvature c. The natural geometric setting of related bi-Hamiltonian structures (Poisson pencils) is the theory of Frobenious manifolds based on the geometry of pencils of contravariant metrics [3].

The other methods of construction of dispersionless systems are based on the application of the quasi-classical limit to the soliton theory. For example, the quasi-classical limit of dressing method is considered by Takasaki and Takebe [4], while the quasi-classical limit of the scalar nonlocal $\bar{\partial}$-problem is presented by Konopelchenko and Alonso [5]; see also the rich literature quoted in these papers.

In the following, we develop an alternative approach to construction of dispersionless systems and related Poisson pencils, based on an R-matrix theory. As is well known, the R-matrix formalism proved very fruitful in systematic construction of soliton systems (see for example [6–8] and the literature quoted therein). So, it seems reasonable to develop such a formalism for dispersionless systems. Recently, important progress in that direction was made by Li [9] who applied the R-matrix theory to Poisson algebras [10]. In this paper, we apply his results to a particular class of Poisson algebras.

The paper is organized as follows. In section 2 we briefly present a number of basic facts and definitions concerning the formalism applied. In section 3 we apply the formalism of the classical R-matrix to the Poisson algebras of formal Laurent series. Then in section 4 we illustrate our results with the known and new $(1+1)$-dimensional integrable dispersionless systems.

2. Hamiltonian dynamics on Lie algebras: *R*-structures

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}^*$ the dual algebra related to $\mathfrak{g}$ by the *duality map* $\langle\cdot,\cdot\rangle \to \mathbb{R}$,

$$\mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R} : (\alpha, a) \mapsto \langle\alpha, a\rangle \qquad a \in \mathfrak{g} \quad \alpha \in \mathfrak{g}^* \tag{2.1}$$

and $\mathcal{D}(\mathfrak{g}^*) := \mathbb{C}^\infty(\mathfrak{g}^*)$ be a space of $\mathbb{C}^\infty$-functions on $\mathfrak{g}^*$. Then, let

$$\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (a, b) \mapsto \mathrm{ad}_a b = [a, b] \qquad a, b \in \mathfrak{g} \tag{2.2}$$

be the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$, i.e. the Lie product, where $[\cdot,\cdot]$ is a Lie bracket on $\mathfrak{g}$. There exists a natural *Lie–Poisson bracket* on $\mathcal{D}(\mathfrak{g}^*)$. Let $F \in \mathcal{D}(\mathfrak{g}^*)$, then a map $\mathrm{d}F : \mathfrak{g} \to \mathfrak{g}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} F(L + tL')\bigg|_{t=0} = \langle L', \mathrm{d}F(L)\rangle \qquad L, L' \in \mathfrak{g}^* \tag{2.3}$$

is a gradient of F. Let $L \in \mathfrak{g}^*$, functions H, F belong to the space of functions on $\mathfrak{g}^* : \mathcal{D}(\mathfrak{g}^*)$, and their gradients $\mathrm{d}H, \mathrm{d}F \in \mathfrak{g}$, then the Lie–Poisson bracket reads

$$\{H, F\}(L) := \langle L, [\mathrm{d}F, \mathrm{d}H]\rangle. \tag{2.4}$$

We confine our further considerations to such algebras $\mathfrak{g}$ for which the dual $\mathfrak{g}^*$ can be identified with $\mathfrak{g}$. So, we assume the existence of a product $(\cdot,\cdot)_\mathfrak{g}$ on $\mathfrak{g}$ which is symmetric, non-degenerate and ad-invariant:

$$(\mathrm{ad}_a b, c)_\mathfrak{g} + (b, \mathrm{ad}_a c)_\mathfrak{g} = 0 \qquad a, b, c \in \mathfrak{g}. \tag{2.5}$$

Then, we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ ($\mathfrak{g}^* \cong \mathfrak{g}$) by setting

$$\langle\alpha, b\rangle = (a, b)_\mathfrak{g} \qquad a, b \in \mathfrak{g} \quad \alpha \in \mathfrak{g}^* \tag{2.6}$$

where $\alpha \in \mathfrak{g}^*$ is identified with $a \in \mathfrak{g}$. Now, we can write the Lie–Poisson bracket as

$$\begin{aligned}\{H, F\}(L) &= \langle L, [\mathrm{d}F, \mathrm{d}H]\rangle = (L, [\mathrm{d}F, \mathrm{d}H])_\mathfrak{g} \\ &= (\mathrm{d}F, [\mathrm{d}H, L])_\mathfrak{g} = (\mathrm{d}F, -\mathrm{ad}_L\, \mathrm{d}H)_\mathfrak{g} \equiv (\mathrm{d}F, \theta(L)\, \mathrm{d}H)_\mathfrak{g}\end{aligned} \tag{2.7}$$

where θ is a Poisson tensor $\theta : \mathfrak{g} \to \mathfrak{g}^*$. Hence, the Hamiltonian dynamical system on $\mathfrak{g}^*$ can be defined by the equation

$$L_t = \theta(L)\,\mathrm{d}H = -\mathrm{ad}_L\,\mathrm{d}H = [\mathrm{d}H, L]. \tag{2.8}$$

Now, we can identify the dynamic equation (2.8) and the Lax equation with a natural Hamiltonian structure

$$L_t = [A, L] = \theta(L)\,\mathrm{d}H = [\mathrm{d}H, L]. \tag{2.9}$$

This abstract approach to integrable systems profits from a deeper understanding of the nature of integrability as well as equips us with a very general and efficient tool for the construction of multi-Hamiltonian systems from scratch.

Definition 2.1. *An R-structure is a Lie algebra $\mathfrak{g}$ equipped with a linear map $R : \mathfrak{g} \to \mathfrak{g}$ (called the classical R-matrix) such that the bracket*

$$[a, b]_R := [Ra, b] + [a, Rb] \qquad a, b \in \mathfrak{g} \tag{2.10}$$

is a second Lie product on $\mathfrak{g}$.

Definition 2.2. *Let A be a commutative, associative algebra with unit 1. If there is a Lie bracket on A such that for each element $a \in A$, the operator* $\mathrm{ad}_a : b \mapsto [a, b]$ *is a derivation of the multiplication, then* $(A, [\cdot, \cdot])$ *is called a* Poisson algebra.

Thus the Poisson algebras are Lie algebras with an additional associative algebra structure (with commutative multiplication and unit 1) related by the derivation property to the Lie bracket.

Theorem 2.3 ([9]). *Let A be a Poisson algebra with Lie bracket $[\cdot, \cdot]$ and non-degenerate ad-invariant pairing $(\cdot, \cdot)_A$ with respect to which the operation of multiplication is symmetric, i.e. $(ab, c)_A = (a, bc)_A, \forall a, b, c \in A$. Assume $R \in \mathrm{End}(A)$ is a classical R-matrix, then for each integer $n \geqslant -1$, the formula*

$$\{H, F\}_n = (L, [R(L^{n+1}\,\mathrm{d}F), \mathrm{d}H] + [\mathrm{d}F, R(L^{n+1}\,\mathrm{d}H)])_A \tag{2.11}$$

where H, F are smooth functions on A, defines a Poisson structure on A. Moreover, all $\{\cdot, \cdot\}_n$ are compatible.

The related Poisson bivectors π_n are given by the following Poisson maps,

$$\pi_n : \mathrm{d}H \mapsto -\mathrm{ad}_L R(L^{n+1}\,\mathrm{d}H) - L^{n+1} R^*(\mathrm{ad}_L\,\mathrm{d}H) \qquad n \geqslant -1 \tag{2.12}$$

where the adjoint of R is defined by the relation

$$(a, Rb)_A = (R^* a, b)_A. \tag{2.13}$$

Note that the bracket (2.11) with $n = -1$ is just a Lie–Poisson bracket with respect to Lie bracket (2.10),

$$\{H, F\}_{-1} = (L, [\mathrm{d}F, \mathrm{d}H]_R)_A. \tag{2.14}$$

We will look for a natural set of functions in involution w.r.t. the Poisson brackets (2.11). A smooth function F on A is ad-invariant if $\mathrm{d}F \in \ker \mathrm{ad}_L$, i.e

$$[\mathrm{d}F, L] = 0 \qquad L \in A \tag{2.15}$$

which are Casimir functionals of the natural Lie–Poisson bracket (2.4).

Hence, the following lemma is valid.

Lemma 2.4 ([9]). *Smooth functions on A which are ad-invariant commute in $\{\cdot,\cdot\}_n$. The Hamiltonian system generated by a smooth ad-invariant function $C(L)$ and the Poisson structure $\{\cdot,\cdot\}_n$ is given by the Lax equation*

$$L_t = [R(L^{n+1}\,\mathrm{d}C), L] \qquad L \in A. \tag{2.16}$$

Let us assume that an appropriate product on Poisson algebra A is given by *the trace form* $\mathrm{Tr} : A \to \mathbb{R}$,

$$(a, b)_A = \mathrm{Tr}(ab) \qquad a, b \in A. \tag{2.17}$$

As we have assumed a non-degenerate trace form Tr on A, we will consider the most natural Casimir functionals given by the trace of powers of L, i.e.

$$C_q(L) = \frac{1}{q+1}\mathrm{Tr}(L^{q+1}). \tag{2.18}$$

The related gradients by (2.3) are of the form

$$\mathrm{d}C_q(L) = L^q. \tag{2.19}$$

Then taking these $C_q(L)$ as Hamiltonian functions, one finds a hierarchy of evolution equations which are multi-Hamiltonian dynamical systems,

$$\begin{aligned} L_{t_q} &= [R(\mathrm{d}C_q), L] \\ &= \pi_{-1}(\mathrm{d}C_q) = \pi_0(\mathrm{d}C_{q-1}) = \cdots = \pi_l(\mathrm{d}C_{q-l-1}) = \cdots. \end{aligned} \tag{2.20}$$

For any R-matrix, both the evolution equations in the hierarchy (2.20) commute due to the involutivity of the Casimir functions C_q. Each equation admits all the Casimir functions as a set of conserved quantities in involution. In this sense, we will regard (2.20) as a hierarchy of *integrable* evolution equations.

To construct the simplest R-structure, let us assume that the Poisson algebra A can be split into a direct sum of Lie subalgebras A_+ and A_-, i.e.

$$A = A_+ \oplus A_- \qquad [A_\pm, A_\pm] \subset A_\pm. \tag{2.21}$$

Denoting the projections onto these subalgebras by $P_\pm$, we define the R-matrix as

$$R = \tfrac{1}{2}(P_+ - P_-) \tag{2.22}$$

which is well defined.

Following the above scheme, we are able to construct in a systematic way integrable multi-Hamiltonian dispersionless systems, with infinite hierarchy of involutive constants of motion and infinite hierarchy of related commuting symmetries, once we fix a Poisson algebra.

3. Poisson algebras of formal Laurent series

Let A be an algebra of Laurent series with respect to p [11],

$$A = \left\{ L = \sum_{i\in\mathbb{Z}} u_i(x) p^i \right\} \tag{3.1}$$

where the coefficients $u_i(x)$ are smooth functions. It is obviously commutative and associative algebra under multiplication. The Lie-bracket can be introduced in infinitely many ways as

$$[L_1, L_2] = p^r\left(\frac{\partial L_1}{\partial p}\frac{\partial L_2}{\partial x} - \frac{\partial L_1}{\partial x}\frac{\partial L_2}{\partial p}\right) := \{L_1, L_2\}_r \qquad r \in \mathbb{Z} \tag{3.2}$$

as $\mathrm{ad}_L = p^r\left(\frac{\partial L}{\partial p}\frac{\partial}{\partial x} - \frac{\partial L}{\partial x}\frac{\partial}{\partial p}\right)$ is a derivation of the multiplication, so $A_r := (A, \{\cdot,\cdot\}_r)$ are Poisson algebras.

Lemma 3.1. *An appropriate symmetric product on A_r is given by a trace form $(a,b)_A := \mathrm{Tr}(ab)$,*

$$\mathrm{Tr}\, L = \int_\Omega \mathrm{res}_r L\, \mathrm{d}x \qquad \mathrm{res}_r L = u_{r-1}(x) \tag{3.3}$$

which is ad*-invariant. In expression (3.3) the integration denotes the equivalence class of differential expressions modulo total derivatives.*

Proof. We assume that $\Omega = \mathbb{S}^1$ if u is periodic or $\Omega = \mathbb{R}$ if u belongs to the Schwartz space. The symmetry is obvious as $L_1L_2 = L_2L_1$. Let $L_1, L_2 \in A : L_1 = \sum_i a_i p^i$, $L_2 = \sum_j b_j p^j$, then

$$\mathrm{res}_r[L_1, L_2] = \mathrm{res}_r\left(p^r \sum_{i,j}(i a_i (b_j)_x - j(a_i)_x b_j)p^{i+j-1}\right) = \sum_i i(a_i b_{-i})_x. \tag{3.4}$$

So, $\mathrm{Tr}[L_1, L_2] = 0$ and hence

$$\begin{aligned}&\mathrm{Tr}([A,B]C) + \mathrm{Tr}(B[A,C])\\ &\qquad = \mathrm{Tr}([A,BC] - B[A,C]) + \mathrm{Tr}(B[A,C]) = \mathrm{Tr}[A,BC] = 0.\end{aligned}$$

□

For a given functional $F(L) = \int_\Omega f(u)\,\mathrm{d}x$, we define its gradient as

$$\mathrm{d}F = \frac{\delta F}{\delta L} = \sum_i \frac{\delta f}{\delta u_i} p^{r-1-i} \tag{3.5}$$

where $\delta f/\delta u_i$ is a variational derivative.

We construct the simplest R-matrix, through a decomposition of A into a direct sum of Lie subalgebras. For a fixed r let

$$\begin{aligned} A_{\geqslant -r+k} &= P_{\geqslant -r+k}A = \left\{L = \sum_{i\geqslant -r+k} u_i(x)p^i\right\}\\ A_{<-r+k} &= P_{<-r+k}A = \left\{L = \sum_{i<-r+k} u_i(x)p^i\right\}\end{aligned} \tag{3.6}$$

where P are appropriate projections.

Proposition 3.2. *$A_{\geqslant -r+k}$, $A_{<-r+k}$ are Lie subalgebras in the following cases:*

(1) $k = 0, r = 0$;
(2) $k = 1, 2, r \in \mathbb{Z}$.

The proof follows from a simple inspection. Then, the R-matrix is given by the projections

$$R = \tfrac{1}{2}(P_{\geqslant -r+k} - P_{<-r+k}) = P_{\geqslant -r+k} - \tfrac{1}{2} = \tfrac{1}{2} - P_{<-r+k}. \tag{3.7}$$

To find R^* one has to find $P^*_{\geqslant -r+k}$ and $P^*_{<-r+k}$ given by the orthogonality relations

$$(P^*_{\geqslant -r+k}, P_{<-r+k}) = (P^*_{<-r+k}, P_{\geqslant -r+k}) = 0. \tag{3.8}$$

So, we have

$$P^*_{\geqslant -r+k} = P_{<2r-k} \qquad P^*_{<-r+k} = P_{\geqslant 2r-k} \tag{3.9}$$

and then

$$R^* = \tfrac{1}{2}(P^*_{\geqslant -r+k} - P^*_{<-r+k}) = \tfrac{1}{2} - P_{\geqslant 2r-k} = P_{<2r-k} - \tfrac{1}{2}. \tag{3.10}$$

Hence, the hierarchy of evolution equations (2.20) for Casimir functionals $C(L)$ with R-matrix given by (3.7) has the form of two equivalent representations

$$L_{t_q} = \{(L^q)_{\geqslant -r+k}, L\}_r = -\{(L^q)_{<-r+k}, L\}_r \qquad L \in A \tag{3.11}$$

which are Lax hierarchies.

We have to explain what type of Lax operators can be used in (3.11) to obtain a consistent operator evolution equivalent to some nonlinear integrable equation. Here, we are interested in extracting closed systems for a finite number of fields. The case of infinite number of fields was considered recently in [11]. Hence, we start by looking for Lax operators L in the general form

$$L = u_N p^N + u_{N-1} p^{N-1} + \cdots + u_{-m+1} p^{-m+1} + u_{-m} p^{-m} \tag{3.12}$$

of Nth order, parametrized by finite number of fields u_i. To obtain a consistent Lax equation, the Lax operator (3.12) has to form a proper submanifold of the full Poisson algebra under consideration, i.e. the left- and right-hand sides of expression (3.11) have to lie inside this submanifold.

Observing (3.11) with some $(L^q)_{<-r+k} = a_{-r+k-1} p^{-r+k-1} + a_{-r+k-2} p^{-r+k-2} + \cdots$ one immediately obtains the highest order of the right-hand side of Lax equation as

$$\begin{aligned} L_t &= (u_N)_t p^N + (u_{N-1})_t p^{N-1} + \cdots \\ &= -\{(L^q)_{<-r+k}, u_N p^N + \text{lower}\}_r \\ &= -((-r+k-1)a_{-r+k-1}(u_N)_x - N(a_{-r+k-1})_x u_N) p^{N+k-2} + \text{lower} \end{aligned} \tag{3.13}$$

where 'lower' represents lower orders. Observing (3.11) with some $(L^q)_{\geqslant -r+k} = \cdots + a_{-r+k+1} p^{-r+k+1} + a_{-r+k} p^{-r+k}$ one immediately obtains the lowest order of the right-hand side of Lax equation (3.11) as

$$\begin{aligned} L_t &= \cdots + (u_{-m+1})_t p^{-m+1} + (u_{-m})_t p^{-m} \\ &= \{(L^q)_{\geqslant -r+k}, \text{higher} + u_{-m} p^{-m}\}_r \\ &= \text{higher} + ((-r+k)a_{-r+k}(u_{-m})_x - (-m)(a_{-r+k})_x u_{-m}) p^{-m+k-1} \end{aligned} \tag{3.14}$$

where 'higher' represents higher orders. Simple consideration of (3.13) and (3.14) with the condition $N \geqslant -m$ leads to the admissible Lax polynomials with a finite number of field coordinates, which form proper submanifolds of Poisson subalgebras. They are given in the form

$$k = 0 \quad r = 0 \qquad L = c_N p^N + c_{N-1} p^{N-1} + u_{N-2} p^{N-2} + \cdots + u_1 p + u_0 \tag{3.15}$$

$$k = 1 \quad r \in \mathbb{Z} \qquad L = c_N p^N + u_{N-1} p^{N-1} + \cdots + u_{1-m} p^{1-m} + u_{-m} p^{-m} \tag{3.16}$$

$$k = 2 \quad r \in \mathbb{Z} \qquad L = u_N p^N + u_{N-1} p^{N-1} + \cdots + u_{1-m} p^{1-m} + c_{-m} p^{-m} \tag{3.17}$$

where the u_i are dynamical fields and c_N, c_{N-1}, c_{-m} are arbitrary time-independent functions of x.

Once we know the restricted Lax operators $L \in A$, we can investigate the form of gradients of Casimir functionals given by powers of L, as well as investigating some further simplest admissible reductions of Lax operators.

In general, the fractional powers of L will lead to interesting results. Let L be given by (3.12), then we consider polynomials of the form

$$\begin{aligned} L^{\frac{1}{N}} &= a_1 p + a_0 + a_{-1} p^{-1} + \cdots && \text{for} \quad N \in \mathbb{Z}_+ \\ L^{\frac{1}{N}} &= a_{-1} p^{-1} + a_{-2} p^{-2} + a_{-3} p^{-3} + \cdots && \text{for} \quad N \in \mathbb{Z}_- \\ L^{\frac{1}{m}} &= \cdots + b_1 p + b_0 + b_{-1} p^{-1} && \text{for} \quad m \in \mathbb{Z}_+ \\ L^{\frac{1}{m}} &= \cdots + b_3 p^3 + b_2 p^2 + b_1 p && \text{for} \quad m \in \mathbb{Z}_- \end{aligned} \tag{3.18}$$

where the coefficients a_i and b_i are obtained by requiring $(L^{\frac{1}{N}})^N = L$ and $(L^{\frac{1}{m}})^m = L$, successively via the recurrent procedure. Therefore, one finds the formal expansion of $L^{\frac{1}{N}}$ and $L^{\frac{1}{m}}$ and can calculate the fractional powers of L for integer q: $L^{\frac{q}{N}}$ and $L^{\frac{q}{m}}$. Note that they are in the form of infinite series, except for the case of integer powers, obviously. In fact, we need only their finite parts $(L^{\frac{q}{N}})_{\geqslant -r+k}$ or $(L^{\frac{q}{m}})_{<-r+k}$. Hence, for a given $L \in A$ in principle we can construct two different hierarchies of Lax equations (3.11).

The case of $k = 0$. Let us consider Lax operators of the form (3.15). One can see that $L^{\frac{q}{N}}$ has the form

$$L^{\frac{q}{N}} = \alpha_q p^q + \alpha_{q-1} p^{q-1} + a_{q-2} p^{q-2} + a_{q-3} p^{q-3} + \text{lower} \qquad q \in \mathbb{Z}_+ \tag{3.19}$$

where α_i, α_{i-1} are arbitrary x-independent functions. The second form $L^{\frac{1}{m}}$, since $m = 0$, gives only the integer powers of L, such that $(L^q)_{\geqslant 0} = L^q$, leading to trivial dynamics $L_t = \{L^q, L\}_0 = 0$. Hence, for $k = 0$ there is only one Lax hierarchy for gradients of Casimir functionals (3.19). There are no further reductions.

The case of $k = 1$. Let us consider Lax operators of the form (3.16). One can see that $L^{\frac{q}{N}}$ and $L^{\frac{q}{m}}$ have the forms

$$L^{\frac{q}{N}} = \alpha_q p^q + a_{q-1} p^{q-1} + a_{q-2} p^{q-2} + a_{q-3} p^{q-3} + \text{lower} \qquad q \in \mathbb{Z}_+ \tag{3.20}$$

$$L^{\frac{q}{m}} = \text{higher} + a_{3-q} p^{3-q} + a_{2-q} p^{2-q} + a_{1-q} p^{1-q} + u_{-m}^{\frac{q}{m}} p^{-q} \qquad q \in \mathbb{Z}_+ \tag{3.21}$$

where α_i is an arbitrary x-independent function. We remark that there is always a further admissible reduction of equations (3.11) given by $u_{-m} = 0$, since such a reduced Lax polynomial would still be of the form (3.16). After such reduction, we have to look for the form of gradients of Casimir functionals. By putting $u_{-m} = 0$ in (3.20), it preserves the order of highest terms and the form. For (3.21) the lowest order disappears, and as all other terms depend linearly on the powers of u_{-m}, such an $L^{\frac{q}{m}}$ will reduce to zero, except for the case $q = m$. We can continue the reductions by putting $u_{1-m} = 0$ and so on. Therefore, the reductions are proper in general only for the gradients of Casimir functionals in the form (3.20).

The case of $k = 2$. Let us consider Lax operators of the form (3.17). One can see that $L^{\frac{q}{N}}$ and $L^{\frac{q}{m}}$ take the form

$$L^{\frac{q}{N}} = u_N^{\frac{q}{N}} p^q + a_{q-1} p^{q-1} + a_{q-2} p^{q-2} + a_{q-3} p^{q-3} + \text{lower} \qquad q \in \mathbb{Z}_+ \tag{3.22}$$

$$L^{\frac{q}{m}} = \text{higher} + a_{3-q} p^{3-q} + a_{2-q} p^{2-q} + a_{1-q} p^{1-q} + \alpha_{-q} p^{-q} \qquad q \in \mathbb{Z}_+ \tag{3.23}$$

where α_i is an arbitrary x-independent function. We remark that there is always a further admissible reduction of equations (3.11) given by $u_N = 0$, since such a reduced Lax polynomial would still be of the form (3.17). The next reduction is $u_{N-1} = 0$ and so on. By analogous considerations as for $k = 1$, these reductions are proper in general only for the gradients of Casimir functionals in the form (3.23).

The different schemes are interrelated as explained in the following theorem.

Theorem 3.3. *Under the transformation*

$$x' = x \qquad p' = p^{-1} \qquad t' = t \tag{3.24}$$

the Lax hierarchy defined by $k = 1, r$ *and* L *transforms into the Lax hierarchy defined by* $k = 2, r' = 2 - r$ *and* L', *i.e.*

$$k = 1, r, L \quad \Longleftrightarrow \quad k = 2, r' = 2 - r, L'. \tag{3.25}$$

Proof. It is readily seen that the Lax operators for $k = 1$ and r of the forms (3.16) transform into the well-restricted Lax operators for $k = 2$ and $r' = 2 - r$ of the forms (3.17). Let us observe that

$$\{A, B\}_r = p^r \left(\frac{\partial A}{\partial p} \frac{\partial B}{\partial x} - \frac{\partial A}{\partial x} \frac{\partial B}{\partial p} \right) = -p'^{-r+2} \left(\frac{\partial A'}{\partial p'} \frac{\partial B'}{\partial x'} - \frac{\partial A'}{\partial x'} \frac{\partial B'}{\partial p'} \right) = -\{A', B'\}'_{r'}$$

and

$$(\mathrm{d}C)'_{\geqslant s} = (\mathrm{d}C')_{\leqslant -s}.$$

Hence, we have

$$\begin{aligned} L_t &= \{(\mathrm{d}C)_{\geqslant -r+1}, L\}_r = -\{(\mathrm{d}C)'_{\geqslant -r+1}, L'\}'_{r'} \\ &= -\{(\mathrm{d}C')_{\leqslant r-1}, L'\}'_{r'} = -\{(\mathrm{d}C')_{<-r'+2}, L'\}'_{r'} = L'_{t'}. \end{aligned}$$

□

Therefore, some dispersionless systems can be reconstructed from different Poisson algebras. Moreover, we remark that the gradients of Casimir functionals for $k = 1$ of the form (3.20), (3.21) by $p' = p^{-1}$ transform into (3.23), (3.22) for $k = 2$, respectively, at a slant.

Two equivalent representations of Poisson bivectors (2.12) with the R-matrix given by (3.7) are defined through the following Poisson maps:

$$\begin{aligned} \pi_n \, \mathrm{d}H &= \{(L^{n+1}\, \mathrm{d}H)_{\geqslant -r+k}, L\}_r + L^{n+1}(\{L, \mathrm{d}H\}_r)_{\geqslant 2r-k} \\ &= -\{(L^{n+1}\, \mathrm{d}H)_{<-r+k}, L\}_r - L^{n+1}(\{L, \mathrm{d}H\}_r)_{<2r-k} \qquad n \geqslant -1. \end{aligned} \tag{3.26}$$

It turns out that the first representation yields direct access to the lowest polynomial order of $\pi_n \, \mathrm{d}H$, whereas the second representation yields information about the highest orders present. There are two options. The best situation is when a given Lax operator forms a proper submanifold of the full Poisson algebra, i.e. the image of the Poisson operator π_n lies in the space tangent to this submanifold for each element. If this is not the case, *the Dirac reduction* can be invoked for restriction of a given Poisson tensor to a suitable submanifold.

The case of $k = 0$. Let us consider the simplest admissible Lax polynomial (3.15) of the form

$$L = p^N + u_{N-2} p^{N-2} + \cdots + u_1 p + u_0. \tag{3.27}$$

This is the well-known dispersionless Gelfand–Dickey case. Then, the gradient of the functional $H(L)$ is given in the form

$$\frac{\delta H}{\delta L} = \frac{\delta H}{\delta u_0} p^{-1} + \frac{\delta H}{\delta u_1} p^{-2} + \cdots + \frac{\delta H}{\delta u_{N-2}} p^{1-N}. \tag{3.28}$$

Observing (3.26) for $n = -1$ one immediately obtains the highest and lowest order of $\pi_{-1}\,\mathrm{d}H$ as

$$\pi_{-1}\left(\frac{\delta H}{\delta L}\right) = \alpha_{N-2}p^{N-2} + \alpha_{N-3}p^{N-3} + \cdots + \alpha_1 p + \alpha_0. \tag{3.29}$$

Hence $\pi_{-1}\,\mathrm{d}H$ is tangent to the submanifold formed by the Lax operator of the form (3.27), and the linear Poisson structure, since $\left(\frac{\delta H}{\delta L}\right)_{\geqslant 0} = 0$, is given by

$$\pi_{-1}\left(\frac{\delta H}{\delta L}\right) = \left(\left\{L, \frac{\delta H}{\delta L}\right\}_0\right)_{\geqslant 0}. \tag{3.30}$$

For $n = 0$, L does not define a proper Poisson submanifold, as

$$\pi_0\left(\frac{\delta H}{\delta L}\right) = \alpha_{N-1}p^{N-1} + \alpha_{N-2}p^{N-2} + \cdots + \alpha_1 p + \alpha_0$$

and a Dirac reduction is required. Let

$$\overline{L} = p^N + up^{N-1} + u_{N-2}p^{N-2} + \cdots + u_0 = L + up^{N-1} \tag{3.31}$$

be an extended Lax polynomial and we shall consider the π_0 Hamiltonian flow for $\overline{L}$ together with the constraint $u = 0$. However, imposition of such a constraint leads to the modification of the π_0 Poisson structure due to the Dirac reduction. We briefly recall the calculation procedure in the example considered. The Hamiltonian flow for u, given by the coefficient of p^{N-1} in the Hamiltonian equation for $\overline{L}$, under the constraint $u = 0$, gives the relation

$$u_t|_{u=0} = \left(\mathrm{res}_0\left\{\frac{\delta H}{\delta \overline{L}}, \overline{L}\right\}_0\right)_{u=0} = 0 \tag{3.32}$$

where

$$\frac{\delta H}{\delta \overline{L}} = \frac{\delta H}{\delta L} + \frac{\delta H}{\delta u}p^{-N}. \tag{3.33}$$

Then solving (3.32) with respect to $\frac{\delta H}{\delta u}$ one gets

$$\frac{\delta H}{\delta u} = -\frac{1}{N}\partial_x^{-1}\,\mathrm{res}_0\left\{L, \frac{\delta H}{\delta L}\right\}_0. \tag{3.34}$$

It means that the function $\frac{\delta H}{\delta u}$ can be expressed in terms of $\frac{\delta H}{\delta u_i}$. This implies

$$\begin{aligned}
\pi_0^{\mathrm{red}}\left(\frac{\delta H}{\delta L}\right) &\equiv \pi_0\left(\frac{\delta H}{\delta \overline{L}}\right)_{u=0} = \left\{\left(L\frac{\delta H}{\delta \overline{L}}\right)_{\geqslant 0}, L\right\}_0 + L\left(\left\{L, \frac{\delta H}{\delta \overline{L}}\right\}_0\right)_{\geqslant 0} \\
&= \left\{\left(L\frac{\delta H}{\delta L} + L\frac{\delta H}{\delta u}p^{-N}\right)_{\geqslant 0}, L\right\}_0 + L\left(\left\{L, \frac{\delta H}{\delta L} + \frac{\delta H}{\delta u}p^{-N}\right\}_0\right)_{\geqslant 0} \\
&= \left\{\left(L\frac{\delta H}{\delta L}\right)_{\geqslant 0}, L\right\}_0 + \left\{\frac{\delta H}{\delta u}, L\right\}_0 + L\left(\left\{L, \frac{\delta H}{\delta L}\right\}_0\right)_{\geqslant 0} \\
&= \left\{\left(L\frac{\delta H}{\delta L}\right)_{\geqslant 0}, L\right\}_0 + L\left(\left\{L, \frac{\delta H}{\delta L}\right\}_0\right)_{\geqslant 0} + \frac{1}{N}\left\{L, \partial_x^{-1}\mathrm{res}_0\left\{L, \frac{\delta H}{\delta L}\right\}_0\right\}_0
\end{aligned} \tag{3.35}$$

i.e. the second Poisson map of dispersionless Gelfand–Dickey systems. Poisson structures π_{-1} and π_0^{red} were constructed for the first time in [12] as the dispersionless limit of the Poisson structures of the Gelfand–Dickey soliton systems. Note that π_0^{red} is purely differential due to the property (3.4).

Observing (3.26) for $n \geqslant 1$ one obtains the highest and lowest orders of $\pi_n\, \mathrm{d}H$ as

$$\pi_n\left(\frac{\delta H}{\delta L}\right) = \alpha_{(n+1)N-1}p^{(n+1)N-1} + \alpha_{(n+1)N-2}p^{(n+1)N-2} + \cdots + \alpha_1 p + \alpha_0. \tag{3.36}$$

Hence, the polynomials of the form (3.27) do not form a proper Poisson submanifold. In fact there is no obvious proper Poisson submanifold for π_n with $n \geqslant 1$, apart from the trivial case of the first-order polynomials with $n = 1$. Nevertheless, the Dirac reduction can be invoked to restrict the bivectors π_n on the polynomials to the submanifold of the form (3.26).

The case of $k = 1$. This case contains new results. Let us consider the simplest admissible Lax polynomial (3.16) of the form

$$L = p^N + u_{N-1}p^{N-1} + \cdots + u_{1-m}p^{1-m} + u_{-m}p^{-m}. \tag{3.37}$$

Then the gradient of the functional $H(L)$ is given in the form

$$\frac{\delta H}{\delta L} = \frac{\delta H}{\delta u_{-m}}p^{r+m-1} + \frac{\delta H}{\delta u_{-m+1}}p^{r+m-2} + \cdots + \frac{\delta H}{\delta u_{N-1}}p^{r-N}. \tag{3.38}$$

Observing (3.26) for $n = -1$ one obtains the highest and lowest orders of $\pi_{-1}\left(\frac{\delta H}{\delta L}\right)$ as

$$\begin{aligned}\pi_{-1}\left(\frac{\delta H}{\delta L}\right) &= ((\ldots)p^{N-1} + \text{lower}) + ((\ldots)p^{2r-2} + \text{lower})\\ &= (\text{higher} + (\ldots)p^{-m}) + (\text{higher} + (\ldots)p^{2r-1})\end{aligned}$$

where 'lower' ('higher') represents lower (higher) orders. Hence, the Lax operators of the type (3.37) form a proper submanifold for $N \geqslant 2r-1 \geqslant -m$, as then $\pi_{-1}\left(\frac{\delta H}{\delta L}\right)$ is tangent to this submanifold. So the linear Poisson map is

$$\pi_{-1}\left(\frac{\delta H}{\delta L}\right) = \left\{\left(\frac{\delta H}{\delta L}\right)_{\geqslant -r+1}, L\right\}_r + \left(\left\{L, \frac{\delta H}{\delta L}\right\}_r\right)_{\geqslant 2r-1}. \tag{3.39}$$

Otherwise a Dirac reduction is required.

For the second Poisson map with $n = 0$, L does not define a proper Poisson submanifold and two distinct cases have to be considered.

$2r - 1 \geqslant 1$:

$$\pi_0\left(\frac{\delta H}{\delta L}\right) = (\ldots)p^{(N-1)+(2r-1)} + \cdots + (\ldots)p^{N-1} + \cdots + (\ldots)p^{-m}$$

hence L is not properly defined and a Dirac reduction is required for additional higher order terms. The simplest case is $r = 1$ with one-field reduction. Let

$$\overline{L} = up^N + u_{N-1}p^{N-1} + u_{N-1}p^{N-2} + \cdots + u_{1-m}p^{1-m} + u_{-m}p^{-m}.$$

The Dirac reduction with the constraint $u = 1$ leads to the second Poisson map in the form

$$\pi_0^{\mathrm{red}}\left(\frac{\delta H}{\delta L}\right) = \left\{\left(L\frac{\delta H}{\delta L}\right)_{\geqslant 0}, L\right\}_1 + L\left(\left\{L, \frac{\delta H}{\delta L}\right\}_1\right)_{\geqslant 1} + \frac{1}{N}\left\{L, \partial_x^{-1}\operatorname{res}_1\left\{L, \frac{\delta H}{\delta L}\right\}_1\right\}_1 \tag{3.40}$$

which is purely differential.

$2r - 1 < 0$:

$$\pi_0\left(\frac{\delta H}{\delta L}\right) = (\dots)p^{N-1} + \dots + (\dots)p^{-m} + \dots + (\dots)p^{-m+(2r-1)}$$

hence L is not properly defined and a Dirac reduction is required for additional lower order terms. The simplest case is $r = 0$ with one-field reduction. Let

$$\overline{L} = p^N + u_{N-1}p^{N-1} + \dots + u_{1-m}p^{1-m} + u_{-m}p^{-m} + up^{-m-1}.$$

The Dirac reduction with the constraint $u = 0$ leads to the second Poisson map in the form

$$\pi_0^{\text{red}}\left(\frac{\delta H}{\delta L}\right) = \left\{\left(L\frac{\delta H}{\delta L}\right)_{\geqslant 1}, L\right\}_0 + L\left(\left\{L, \frac{\delta H}{\delta L}\right\}_0\right)_{\geqslant -1} + \frac{1}{m}\left\{L, \partial_x^{-1}\operatorname{res}_0\left\{L, \frac{\delta H}{\delta L}\right\}_0\right\}_0 \tag{3.41}$$

which is purely differential. This special case was considered recently in [13].

Observing (3.26) for $n \geqslant 1$ one obtains the highest and lowest orders of $\pi_n\left(\frac{\delta H}{\delta L}\right)$ as

$$\begin{aligned}\pi_n\left(\frac{\delta H}{\delta L}\right) &= ((\dots)p^{N-1} + \text{lower}) + ((\dots)p^{(n+1)N+2r-2} + \text{lower})\\ &= (\text{higher} + (\dots)p^{-m}) + (\text{higher} + (\dots)p^{-(n+1)m+2r-1})\end{aligned}$$

where ‘lower’ (‘higher’) represents lower (higher) orders. Hence, the Lax operators of the type (3.37) do not form a proper Poisson submanifold for the π_n with $n \geqslant 1$, apart from the trivial case of $N = -m = \frac{1-2r}{n}$. Hence, one has to apply Dirac reduction to restrict the bivectors π_n on the polynomials to the submanifold of the form (3.26).

The case of $k = 2$. This has not been considered yet. Let us consider a Lax polynomial (3.17) of the form

$$L = u_N p^N + u_{N-1}p^{N-1} + \dots + u_{1-m}p^{1-m} + p^{-m}. \tag{3.42}$$

Then, the gradient of functional $H(L)$ is given in the form

$$\frac{\delta H}{\delta L} = \frac{\delta H}{\delta u_{1-m}}p^{r+m-2} + \dots + \frac{\delta H}{\delta u_{N-1}}p^{r-N} + \frac{\delta H}{\delta u_N}p^{r-N-1}. \tag{3.43}$$

Then by analogous consideration as for $k = 1$ or by theorem 3.3, for the first Poisson structure with $n = -1$, L defines a proper Poisson submanifold for $N \geqslant 2r - 3 \geqslant -m$, so the first Poisson map in this case is

$$\pi_{-1}\left(\frac{\delta H}{\delta L}\right) = \left\{\left(\frac{\delta H}{\delta L}\right)_{\geqslant -r+2}, L\right\}_r + \left(\left\{L, \frac{\delta H}{\delta L}\right\}_r\right)_{\geqslant 2r-2}. \tag{3.44}$$

Otherwise a Dirac reduction is required.

For the second Poisson map with $n = 0$, L does not define a proper Poisson submanifold and again two distinct cases have to be considered.

$2r - 3 > 0$:

$$\pi_0\left(\frac{\delta H}{\delta L}\right) = (\dots)p^{N+(2r-3)} + \dots + (\dots)p^N + \dots + (\dots)p^{1-m}$$

hence L is not properly defined and a Dirac reduction is required for additional higher order terms. The simplest case is $r = 2$ with one-field reduction. Let

$$\overline{L} = up^{N+1} + u_N p^N + u_{N-1}p^{N-1} + \dots + u_{1-m}p^{1-m} + p^{-m}$$

then the Dirac reduction with the constraint $u = 0$ leads to the second Poisson map in the form

$$\pi_0^{\mathrm{red}}\left(\frac{\delta H}{\delta L}\right) = \left\{\left(L\frac{\delta H}{\delta L}\right)_{\geqslant 0}, L\right\}_2 + L\left(\left\{L, \frac{\delta H}{\delta L}\right\}_2\right)_{\geqslant 2} + \frac{1}{N}\left\{L, \partial_x^{-1}\operatorname{res}_2\left\{L, \frac{\delta H}{\delta L}\right\}_2\right\}_2 \tag{3.45}$$

which is purely differential.

$2r - 3 < 0:$

$$\pi_0\left(\frac{\delta H}{\delta L}\right) = (\ldots)p^N + \cdots + (\ldots)p^{-m+1} + \cdots + (\ldots)p^{-m+1+(2r-3)}$$

hence L is not properly defined and a Dirac reduction is required for additional lower order terms. The simplest case is $r = 1$ with one-field reduction. Let

$$\overline{L} = u_N p^N + u_{N-1}p^{N-1} + \cdots + u_{2-m}p^{2-m} + u_{1-m}p^{1-m} + up^{-m}$$

then the Dirac reduction with the constraint $u = 1$ leads to the second Poisson map in the form

$$\pi_0^{\mathrm{red}}\left(\frac{\delta H}{\delta L}\right) = \left\{\left(L\frac{\delta H}{\delta L}\right)_{\geqslant 1}, L\right\}_1 + L\left(\left\{L, \frac{\delta H}{\delta L}\right\}_1\right)_{\geqslant 0} + \frac{1}{m}\left\{L, \partial_x^{-1}\operatorname{res}_1\left\{L, \frac{\delta H}{\delta L}\right\}_1\right\}_1 \tag{3.46}$$

which is again purely differential.

Now we present one example of three-field Dirac reduction. Let us consider the case with $r = 0$, then

$$\overline{L} = u_N p^N + \cdots + u_{1-m}p^{1-m} + up^{-m} + vp^{-m-1} + wp^{-m-2}.$$

The Dirac reduction with constraints $u = 1$, $v = w = 0$ gives the following reduced Poisson map,

$$\pi_0^{\mathrm{red}}\left(\frac{\delta H}{\delta L}\right) = \left\{\left(L\frac{\delta H}{\delta L}\right)_{\geqslant 2}, L\right\}_0 + L\left(\left\{L, \frac{\delta H}{\delta L}\right\}_0\right)_{\geqslant -2} + \{L, Ap + B + Cp^{-1}\}_0 \tag{3.47}$$

where

$$\begin{aligned}
C &= \frac{1}{m}\partial_x^{-1}\left(\left\{L, \frac{\delta H}{\delta L}\right\}_0\right)_{-2}\\
B &= \frac{1}{m}\partial_x^{-1}\left(\left\{L, \frac{\delta H}{\delta L}\right\}_0\right)_{-1} + \frac{1}{m}u_{-m+1}C\\
A &= \frac{1}{m}\partial_x^{-1}\operatorname{res}_0\left\{L, \frac{\delta H}{\delta L}\right\}_0 + \frac{1}{m^2}\partial_x^{-1}u_{-m+1}\left(\left\{L, \frac{\delta H}{\delta L}\right\}_0\right)_{-1}\\
&\quad + \frac{1}{m}\left(u_{-m+2} - \frac{1}{2}\frac{m-1}{m}u_{-m+1}^2\right)C + \frac{1}{m}\partial_x^{-1}\left(u_{-m+2} - \frac{1}{2}\frac{m-1}{m}u_{-m+1}^2\right)C_x
\end{aligned}$$

generally nonlocal.

Then by analogous consideration as for $k = 1$ or by theorem 3.3, we see that Lax operators of the form (3.42) do not form a proper Poisson submanifold for the π_n with $n \geqslant 1$, apart from the trivial case of $N = -m = \frac{3-2r}{n}$. Hence, one has to apply the Dirac reduction to restrict the bivectors π_n on the polynomials to the submanifold of the form (3.26).

Hence we know the Poisson structure for $(1+1)$-dispersionless systems constructed from Poisson algebras, and since we are interested in multi-Hamiltonian systems,

$$L_{t_q} = \{(L^q)_{\geqslant -r+k}, L\}_r = \pi_{-1}\,\mathrm{d}H_1 = \pi_0\,\mathrm{d}H_0 = \pi_{-1}\,\mathrm{d}H_{-1} = \cdots \tag{3.48}$$

we shall now consider the problem of their construction. The conserved quantities H_i from (2.18) are defined as follows:

$$H_i(L) = \frac{1}{q+i}\mathrm{Tr}(L^{q+i}) = \frac{1}{q+i}\int_\Omega \mathrm{res}_r\left(L^{q+i}\right)\mathrm{d}x. \tag{3.49}$$

4. A list of some (1 + 1)-dimensional dispersionless systems

In this section, we will display a list of the simplest nonlinear dispersionless integrable systems. Calculating the powers $L^{\frac{n}{N}}$ we consider the Lax hierarchy

$$L_{t_n} = \left\{\left(L^{\frac{n}{N}}\right)_{\geqslant -r+k}, L\right\}_r \qquad n = 1, 2, 3, \ldots. \tag{4.1}$$

The second hierarchy with powers $L^{\frac{n}{m}}$ can be obtained by the transformation from theorem 3.3, which we leave for the interested reader. In general, for simplicity we present only the bi-Hamiltonian structure. For $k = 0$ and $k = 1$ the choice $n = 1 - r$ will always lead to the dynamics $(u_i)_{t_{1-r}} = (1-r)(u_i)_x$ for the fields u_i in L, so that we may identify $t_{1-r} = \frac{1}{1-r}x$ in these cases. For $k = 0$ and integer values of n/N the equations become trivial, because then $(L^{\frac{n}{N}})_{\geqslant 0} = L$. For each choice of $k = 0$, 1 or 2 and N we will exhibit the first nontrivial of the nonlinear Lax equations (4.1) associated with a chosen operator L.

The case of $k = 0$.

Example 4.1. *Dispersionless Korteweg–de Vries*: $k = 0, r = 0, N = 2$.

This is a standard case of the dispersionless Korteweg–de Vries (dKdV) hierarchy. The Lax operator for the dKdV has the form

$$L = p^2 + u. \tag{4.2}$$

We derive the dKdV equation

$$L_{t_3} = \left\{\left(L^{\frac{3}{2}}\right)_{\geqslant 0}, L\right\}_0 \iff u_{t_3} = \frac{3}{2}uu_x = \pi_{-1}\,\mathrm{d}H_1 = \pi_0^{\mathrm{red}}\,\mathrm{d}H_0 = \pi_1^{\mathrm{red}}\,\mathrm{d}H_{-1} \tag{4.3}$$

where we get the Poisson tensors from (3.30) and (3.35),

$$\begin{aligned} &\pi_{-1} = 2\partial_x \qquad \pi_0^{\mathrm{red}} = \partial_x u + u\partial_x \\ &\pi_1^{\mathrm{red}} = \pi_0^{\mathrm{red}}(\pi_{-1})^{-1}\pi_0^{\mathrm{red}} = \partial_x u^2 + u^2\partial_x - \tfrac{1}{2}u_x\partial_x^{-1}u_x \end{aligned} \tag{4.4}$$

and the respective Hamiltonians

$$H_1 = \frac{1}{8}\int_\Omega u^3\,\mathrm{d}x \qquad H_0 = \frac{1}{4}\int_\Omega u^2\,\mathrm{d}x \qquad H_{-1} = \int_\Omega u\,\mathrm{d}x. \tag{4.5}$$

Example 4.2. *Dispersionless Boussinesq*: $k = 0, r = 0, N = 3$.

The Lax operator is given by

$$L = p^3 + up + v. \tag{4.6}$$

We derive

$$L_{t_2} = \left\{\left(L^{\frac{2}{3}}\right)_{\geqslant 0}, L\right\}_0 \iff \begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} 2v_x \\ -\frac{2}{3}uu_x \end{pmatrix} = \pi_{-1}\,\mathrm{d}H_1 = \pi_0^{\mathrm{red}}\,\mathrm{d}H_0. \tag{4.7}$$

Eliminating the field v from this equation, we can derive the $(1+1)$-dimensional dispersionless Boussinesq equation

$$u_{tt} = -\tfrac{2}{3}(u^2)_{xx}. \tag{4.8}$$

The respective Poisson tensors are

$$\pi_{-1} = 3\begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \qquad \pi_0^{\mathrm{red}} = \begin{pmatrix} \partial_x u + u\partial_x & 2\partial_x v + v\partial_x \\ \partial_x v + 2v\partial_x & -\frac{2}{3}u\partial_x u \end{pmatrix} \tag{4.9}$$

and the Hamiltonians are given in the following form:

$$H_1 = \frac{1}{3}\int_\Omega \left(v^2 - \frac{1}{9}u^3\right)\mathrm{d}x \qquad H_0 = \int_\Omega v\,\mathrm{d}x. \tag{4.10}$$

Example 4.3. *The three field case*: $k=0, r=0, N=4$.

The Lax operator is

$$L = p^4 + up^2 + vp + w \tag{4.11}$$

then

$$L_{t_2} = \left\{\left(L^{\frac{2}{4}}\right)_{\geqslant 0}, L\right\}_0 \quad \Longleftrightarrow \quad \begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_2} = \begin{pmatrix} 2v_x \\ -uu_x + 2w_x \\ -\frac{1}{2}u_x v \end{pmatrix} = \pi_{-1}\,\mathrm{d}H_1 = \pi_0^{\mathrm{red}}\,\mathrm{d}H_0 \tag{4.12}$$

where

$$\pi_{-1} = \begin{pmatrix} 0 & 0 & 4\partial_x \\ 0 & 4\partial_x & 0 \\ 4\partial_x & 0 & \partial_x u + u\partial_x \end{pmatrix} \tag{4.13}$$

$$\pi_0^{\mathrm{red}} = \begin{pmatrix} \partial_x u + u\partial_x & 2\partial_x v + v\partial_x & 3\partial_x w + w\partial_x \\ \partial_x v + 2v\partial_x & -u\partial_x u + 2\partial_x w + 2w\partial_x & -\frac{1}{2}u\partial_x v \\ \partial_x w + 3w\partial_x & -\frac{1}{2}v\partial_x u & -\frac{3}{4}v\partial_x v + u\partial_x w + w\partial_x u \end{pmatrix}$$

$$H_1 = \frac{1}{2}\int_\Omega \left(-\frac{1}{4}u^2 v + vw\right)\mathrm{d}x \qquad H_0 = \int_\Omega v\,\mathrm{d}x. \tag{4.14}$$

The case of $k=1$.

Example 4.4. *Three field hierarchy*: $k=1, r \in \mathbb{Z}\setminus\{2\}$.

The Lax operator has the form (3.16), with $N = 2-r, m = r+1$,

$$L = p^{2-r} + up^{1-r} + vp^{-r} + wp^{-r-1}. \tag{4.15}$$

Then we find

$$L_{t_{2-r}} = \left\{(L)_{\geqslant -r+1}, L\right\}_r \quad \Longleftrightarrow \quad \begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_{2-r}} = \begin{pmatrix} (2-r)v_x \\ ru_x v + (1-r)uv_x + (2-r)w_x \\ (1+r)u_x w + (1-r)uw_x \end{pmatrix}. \tag{4.16}$$

This Lax operator forms a proper submanifold as regards π_{-1} only for $r = 0, 1$. Otherwise a Dirac reduction is required. Then for $r = 0$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_2} = \begin{pmatrix} 2v_x \\ uv_x + 2w_x \\ u_x w + uw_x \end{pmatrix} = \pi_{-1}\, \mathrm{d}H_1 = \pi_0^{\mathrm{red}}\, \mathrm{d}H_0 \tag{4.17}$$

where

$$\begin{aligned} \pi_{-1} &= \begin{pmatrix} 0 & 0 & 2\partial_x \\ 0 & 2\partial_x & u\partial_x \\ 2\partial_x & \partial_x u & 0 \end{pmatrix} \\ \pi_0^{\mathrm{red}} &= \begin{pmatrix} 6\partial_x & 4\partial_x u & 2\partial_x v \\ 4u\partial_x & 2u\partial_x u + \partial_x v + v\partial_x & u\partial_x v + 2\partial_x w + w\partial_x \\ 2v\partial_x & v\partial_x u + \partial_x w + 2w\partial_x & u\partial_x w + w\partial_x u \end{pmatrix} \end{aligned} \tag{4.18}$$

$$H_1 = \int_\Omega vw\, \mathrm{d}x \qquad H_0 = \int_\Omega w\, \mathrm{d}x. \tag{4.19}$$

For $r = 1$ we have

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_1} = \begin{pmatrix} v_x \\ u_x v + w_x \\ 2u_x w \end{pmatrix} = \pi_{-1}\, \mathrm{d}H_1 = \pi_0^{\mathrm{red}}\, \mathrm{d}H_0 \tag{4.20}$$

where

$$\begin{aligned} \pi_{-1} &= \begin{pmatrix} 0 & \partial_x v & 2\partial_x w \\ v\partial_x & \partial_x w + w\partial_x & 0 \\ 2w\partial_x & 0 & 0 \end{pmatrix} \\ \pi_0^{\mathrm{red}} &= \begin{pmatrix} \partial_x v + v\partial_x & u\partial_x v + 2\partial_x w + w\partial_x & 2u\partial_x w \\ v\partial_x u + \partial_x w + 2w\partial_x & 2v\partial_x v + u\partial_x w + w\partial_x u & 4v\partial_x w \\ 2w\partial_x u & 4w\partial_x v & 6w\partial_x w \end{pmatrix} \end{aligned} \tag{4.21}$$

$$H_1 = \frac{1}{2}\int_\Omega (u^2 + 2v)\, \mathrm{d}x \qquad H_0 = \int_\Omega u\, \mathrm{d}x. \tag{4.22}$$

Example 4.5. *Dispersionless Toda*: $k = 1, r \in \mathbb{Z} \setminus \{2\}$.

The first admissible reduction $w = 0$ of (4.15) leads to the two-field Lax operator

$$L = p^{2-r} + up^{1-r} + vp^{-r}. \tag{4.23}$$

This Lax operator forms a proper submanifold as regards π_{-1} only for $r = 1$, in other cases a Dirac reduction is required. For $r = 1$ by reduction of (4.20) we get the dispersionless Toda equation

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = \begin{pmatrix} v_x \\ u_x v \end{pmatrix} = \pi_{-1}\, \mathrm{d}H_1 = \pi_0^{\mathrm{red}}\, \mathrm{d}H_0 \tag{4.24}$$

where

$$\pi_{-1} = \begin{pmatrix} 0 & \partial_x v \\ v\partial_x & 0 \end{pmatrix} \qquad \pi_0^{\mathrm{red}} = \begin{pmatrix} \partial_x v + v\partial_x & u\partial_x v \\ v\partial_x u & 2v\partial_x v \end{pmatrix} \tag{4.25}$$

$$H_1 = \frac{1}{2}\int_\Omega (u^2 + 2v)\, \mathrm{d}x \qquad H_0 = \int_\Omega u\, \mathrm{d}x. \tag{4.26}$$

For $r = 0$ we have

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \begin{pmatrix} 2v_x \\ uv_x \end{pmatrix} \tag{4.27}$$

but we lose the bi-Hamiltonian structure since there are no Dirac reductions with the constraint $w = 0$ of (4.18).

The next admissible reduction $w = v = 0$ of (4.16) leads to the non-interesting trivial equation $L_{t_{2-r}} = 0$ since $(L)_{\geqslant -r+1} = L$.

Example 4.6. *Three field hierarchy*: $k = 1, r \in \mathbb{Z} \setminus \{1\}$.

The Lax operator has the form (3.16), with $N = 1 - r, m = r + 2$,

$$L = p^{1-r} + up^{-r} + vp^{-r-1} + wp^{-r-2}. \tag{4.28}$$

Then we find

$$L_{t_{2-r}} = \left\{ \left(L^{\frac{2-r}{1-r}} \right)_{\geqslant -r+1}, L \right\}_r \quad \Longleftrightarrow$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_{2-r}} = \frac{2-r}{1-r} \begin{pmatrix} uu_x + (1-r)v_x \\ (1+r)u_x v + (1-r)uv_x + (1-r)w_x \\ (2+r)u_x w + (1-r)uw_x \end{pmatrix}. \tag{4.29}$$

This Lax operator forms a proper submanifold as regards π_{-1} only for $r = 0$, in other cases a Dirac reduction is required. Then for $r = 0$ we have

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_2} = 2 \begin{pmatrix} uu_x + v_x \\ u_x v + uv_x + w_x \\ 2u_x w + uw_x \end{pmatrix} = \pi_{-1}\, \mathrm{d}H_1 = \pi_0^{\mathrm{red}}\, \mathrm{d}H_0 \tag{4.30}$$

where

$$\pi_{-1} = \begin{pmatrix} 0 & \partial_x & 0 \\ \partial_x & 0 & 0 \\ 0 & 0 & \partial_x w + w\partial_x \end{pmatrix}$$
$$\pi_0^{\mathrm{red}} = \begin{pmatrix} \frac{3}{2}\partial_x & \partial_x u & \frac{1}{2}\partial_x v \\ u\partial_x & \partial_x v + v\partial_x & 2\partial_x w + w\partial_x \\ \frac{1}{2}v\partial_x & \partial_x w + 2w\partial_x & -\frac{1}{2}v\partial_x u + u\partial_x w + w\partial_x u \end{pmatrix} \tag{4.31}$$

$$H_1 = \int_\Omega (u^2 v + v^2 + 2uw)\, \mathrm{d}x \qquad H_0 = \int_\Omega (uv + w)\, \mathrm{d}x. \tag{4.32}$$

Example 4.7. *Benney system*: $k = 1, r \in \mathbb{Z} \setminus \{1\}$.

The first admissible reduction $w = 0$ of (4.28) leads to the two-field Lax operator

$$L = p^{1-r} + up^{-r} + vp^{-r-1}. \tag{4.33}$$

This Lax operator forms a proper submanifold as regards π_{-1} only for $r = 0$, otherwise a Dirac reduction is required. For $r = 0$ by reduction of (4.30) we get the Benney system

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = 2 \begin{pmatrix} uu_x + v_x \\ u_x v + uv_x \end{pmatrix} = \pi_{-1}\, \mathrm{d}H_1 = \pi_0^{\mathrm{red}}\, \mathrm{d}H_0 \tag{4.34}$$

where

$$\pi_{-1} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \qquad \pi_0^{\mathrm{red}} = \begin{pmatrix} 2\partial_x & \partial_x u \\ u\partial_x & \partial_x v + v\partial_x \end{pmatrix} \tag{4.35}$$

$$H_1 = \int_\Omega (u^2 v + v^2)\, \mathrm{d}x \qquad H_0 = \int_\Omega uv\, \mathrm{d}x. \tag{4.36}$$

The next admissible reduction $w = v = 0$ of (4.29) leads to

$$u_{t_{2-r}} = \frac{2-r}{1-r} u u_x \tag{4.37}$$

but for $r = 0$ we lose the bi-Hamiltonian structure since there are no Dirac reductions with $w = 0$ of (4.31).

The case of $k = 2$.

Example 4.8. *Two-field hierarchy*: $k = 2, r \in \mathbb{Z} \setminus \{3\}$.

The Lax operator is given by

$$L = u^{3-r} p^{3-r} + vp^{2-r} + p^{1-r} \tag{4.38}$$

then we have

$$\begin{aligned} L_{t_{4-r}} &= \left\{ \left(L^{\frac{4-r}{3-r}} \right)_{\geqslant -r+2}, L \right\}_r \quad \Longleftrightarrow \\ u_{t_{4-r}} &= \frac{4-r}{2(3-r)^2} (2(3-r)(2r-3)u^{2-r} u_x v \\ &\quad + 2(3-r)u^{3-r} v_x - (2-r)v^2 v_x + (2-r)^2 (\ln u)_x v^3) \\ v_{t_{4-r}} &= \frac{4-r}{2(3-r)^2} (2(r-1)(3-r)^2 u_x + (1-r)(2-r)(3-r)u^{r-3} u_x v^2 \\ &\quad - 2(1-r)(3-r)u^{r-2} v v_x). \end{aligned} \tag{4.39}$$

For $r = 2$ we find

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = 2 \begin{pmatrix} u_x v + u v_x \\ u_x + v v_x \end{pmatrix} \tag{4.40}$$

which is again a Benney system with the known bi-Hamiltonian structure, this time reconstructed from formulae (3.44) and (3.45).

Example 4.9. *Two-field hierarchy*: $k = 2, r \in \mathbb{Z} \setminus \{2\}$.

The Lax operator has the form (3.17), with $N = 2 - r, m = r$,

$$L = u^{2-r} p^{2-r} + vp^{1-r} + p^{-r}. \tag{4.41}$$

Then we find

$$L_{t_{2-r}} = \{(L)_{\geqslant -r+2}, L\}_r \quad \Longleftrightarrow \quad \begin{pmatrix} u \\ v \end{pmatrix}_{t_{2-r}} = \begin{pmatrix} (r-1)r u_x v + u v_x \\ (2-r) r u^{1-r} u_x \end{pmatrix}. \tag{4.42}$$

This Lax operator forms a proper submanifold as regards π_{-1} only for $r = 1$, otherwise a Dirac reduction is required. Then for $r = 1$

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} = \begin{pmatrix} u v_x \\ u_x \end{pmatrix}. \tag{4.43}$$

It is again a Toda system with the known bi-Hamiltonian structure, this time reconstructed from formulae (3.44) and (3.46).

Example 4.10. *Three-field hierarchy*: $k = 2, r \in \mathbb{Z} \setminus \{2\}$.

The Lax operator has the form (3.16), with $N = 2 - r, m = r + 1$,

$$L = up^{2-r} + vp^{1-r} + wp^{-r} + p^{-r-1}. \tag{4.44}$$

Then we find

$$L_{t_{2-r}} = \left\{(L)_{\geqslant -r+2}, L\right\}_r \qquad \Longleftrightarrow \qquad \begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_{2-r}} = \begin{pmatrix} (r-1)u_x v + (2-r)uv_x \\ ru_x w + (2-r)uw_x \\ (1+r)u_x \end{pmatrix}. \tag{4.45}$$

This Lax operator forms a proper submanifold as regards π_{-1} only for $r = 1$, in other cases a Dirac reduction is required. Then for $r = 1$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_1} = \begin{pmatrix} uv_x \\ u_x w + uw_x \\ 2u_x \end{pmatrix} = \pi_{-1}\, \mathrm{d}H_1 = \pi_0^{\mathrm{red}}\, \mathrm{d}H_0 \tag{4.46}$$

where

$$\begin{aligned} \pi_{-1} &= \begin{pmatrix} 0 & u\partial_x & 0 \\ \partial_x u & 0 & 0 \\ 0 & 0 & 2\partial_x \end{pmatrix} \\ \pi_0^{\mathrm{red}} &= \begin{pmatrix} \frac{3}{2}u\partial_x u & u\partial_x v & \frac{1}{2}u\partial_x w \\ v\partial_x u & u\partial_x w + w\partial_x u & \partial_x u + 2u\partial_x \\ \frac{1}{2}w\partial_x u & 2\partial_x u + u\partial_x & -\frac{1}{2}w\partial_x w + \partial_x v + v\partial_x \end{pmatrix} \end{aligned} \tag{4.47}$$

$$H_1 = \frac{1}{2}\int_\Omega (v^2 + 2uw)\, \mathrm{d}x \qquad H_0 = \int_\Omega v\, \mathrm{d}x. \tag{4.48}$$

Example 4.11. *Two-field hierarchy*: $k = 2, r \in \mathbb{Z} \setminus \{1\}$.

The Lax operator has the form (3.17), with $N = 1 - r, m = r + 1$,

$$L = u^{1-r} p^{1-r} + vp^{-r} + p^{-r-1}. \tag{4.49}$$

Then we find

$$L_{t_{2-r}} = \left\{\left(L^{\frac{2-r}{1-r}}\right)_{\geqslant -r+2}, L\right\}_r \qquad \Longleftrightarrow \qquad \begin{pmatrix} u \\ v \end{pmatrix}_{t_{2-r}} = \frac{2-r}{1-r}\begin{pmatrix} ruu_x v + u^2 v_x \\ (1-r^2)u^{1-r}u_x \end{pmatrix}. \tag{4.50}$$

Let us consider the case of $r = 0$. To get π_{-1} we have to make a Dirac reduction as the condition $N \geqslant 2r - 3 \geqslant -m$ is violated. The simplest admissible Lax polynomial has the form

$$\overline{L} = up + v + wp^{-1} + zp^{-2} \tag{4.51}$$

and the Poisson operator reconstructed from (3.44) is

$$\pi_{-1} = \begin{pmatrix} 0 & 0 & 0 & 2u\partial_x - \partial_x u \\ 0 & 0 & u\partial_x & -v_x \\ 0 & \partial_x u & 0 & -\partial_x w \\ 2\partial_x u - u\partial_x & v_x & -w\partial_x & -\partial_x z - z\partial_x \end{pmatrix}. \tag{4.52}$$

Then, reduction of (4.52) with constraints $z = 0$, $w = 1$ gives

$$\pi_{-1}^{\mathrm{red}} = \begin{pmatrix} 0 & u^2\partial_x \\ \partial_x u^2 & 0 \end{pmatrix} \tag{4.53}$$

while the second Poisson operator, constructed from (3.45), takes the form

$$\pi_0^{\mathrm{red}} = \begin{pmatrix} u^2\partial_x u + u\partial_x u^2 & u^2 v_x + u^2\partial_x v \\ -u^2 v_x + v\partial_x u^2 & 2u\partial_x u \end{pmatrix}. \tag{4.54}$$

Fortunately, both Poisson operators are again differential. Hence

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = 2\begin{pmatrix} u^2 v_x \\ u u_x \end{pmatrix} = \pi_{-1}^{\mathrm{red}}\,\mathrm{d}H_1 = \pi_0^{\mathrm{red}}\,\mathrm{d}H_0 \tag{4.55}$$

where

$$H_1 = \int_\Omega (u + v^2)\,\mathrm{d}x \qquad H_0 = \int_\Omega v\,\mathrm{d}x. \tag{4.56}$$

Example 4.12. *Three-field hierarchy*: $k = 2, r \in \mathbb{Z}\setminus\{1\}$.

The Lax operator has the form (3.17), with $N = 1 - r, m = r + 2$,

$$L = u^{1-r}p^{1-r} + vp^{-r} + wp^{-r-1} + p^{-r-2}. \tag{4.57}$$

Then we find

$$\begin{gathered} L_{t_{2-r}} = \left\{\left(L^{\frac{2-r}{1-r}}\right)_{\geqslant -r+2}, L\right\}_r \qquad \Longleftrightarrow \\ \begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_{2-r}} = \frac{2-r}{1-r}\begin{pmatrix} ruu_x v + u^2 v_x \\ (1-r)u^{1-r}((1+r)u_x w + uw_x) \\ (2-r)(1-r)u^{1-r}u_x \end{pmatrix}. \end{gathered} \tag{4.58}$$

Let us consider the case for $r = 0$. Again condition $N \geqslant 2r - 3 \geqslant -m$ is violated but reducing (4.52) with constraint $z = 1$ we get the first Poisson operator in the form

$$\pi_{-1}^{\mathrm{red}} = \begin{pmatrix} \frac{1}{2}u\partial_x u - \frac{1}{2}u_x\partial_x^{-1}u_x & \frac{1}{2}uv_x - \frac{1}{2}u_x\partial_x^{-1}v_x & -\frac{1}{2}uw\partial_x + \frac{1}{2}u_x w - \frac{1}{2}u_x\partial_x^{-1}w_x \\ * & -\frac{1}{2}v_x\partial_x^{-1}v_x & u\partial_x + \frac{1}{2}v_x w - \frac{1}{2}v_x\partial_x^{-1}w_x \\ * & * & \frac{1}{4}w^2\partial_x + \frac{1}{4}\partial_x w^2 - \frac{1}{2}w_x\partial_x^{-1}w_x \end{pmatrix} \tag{4.59}$$

where $*$ denotes the elements that make the matrix skew-adjoint. The second Poisson operator calculated according to (3.45) is

$$\begin{aligned}
\left(\pi_0^{\mathrm{red}}\right)_{11} &= \tfrac{1}{4}u^2\left(v - \tfrac{1}{4}w^2\right)\partial_x + \tfrac{1}{4}\left[u\left(v - \tfrac{1}{4}w^2\right)_x - u_x\left(v - \tfrac{1}{4}w^2\right)\right]\partial_x^{-1}u_x \\
&\quad + \tfrac{1}{4}\partial_x u^2\left(v - \tfrac{1}{4}w^2\right) + \tfrac{1}{4}u_x\partial_x^{-1}\left[u\left(v - \tfrac{1}{4}w^2\right)_x - u_x\left(v - \tfrac{1}{4}w^2\right)\right] \\
\left(\pi_0^{\mathrm{red}}\right)_{12} &= \tfrac{1}{4}u^2 w\partial_x + \tfrac{1}{2}u\left(v - \tfrac{1}{4}w^2\right)v_x + \tfrac{1}{4}\left[u\left(v - \tfrac{1}{4}w^2\right)_x - u_x\left(v - \tfrac{1}{4}w^2\right)\right]\partial_x^{-1}v_x \\
&\quad + \tfrac{1}{4}u_x\partial_x^{-1}\left[uw_x - v_x\left(v - \tfrac{1}{4}w^2\right)\right] \\
\left(\pi_0^{\mathrm{red}}\right)_{13} &= \left(\tfrac{3}{2}u^2 + \tfrac{1}{8}uw^3\right)\partial_x - \tfrac{1}{2}uu_x + \tfrac{1}{4}w\left(2vu_x - uv_x - \tfrac{1}{2}w^2u_x + \tfrac{1}{2}uww_x\right) \\
&\quad + \tfrac{1}{4}\left[u\left(v - \tfrac{1}{4}w^2\right)_x - u_x\left(v - \tfrac{1}{4}w^2\right)\right]\partial_x^{-1}w_x - \tfrac{1}{2}uvw\partial_x \\
&\quad + \tfrac{1}{4}u_x\partial_x^{-1}\left[2u_x - (vw)_x + \tfrac{1}{4}(w^3)_x\right] \\
\left(\pi_0^{\mathrm{red}}\right)_{22} &= \tfrac{3}{2}u\partial_x u + \tfrac{1}{4}\left[uw_x - \left(v - \tfrac{1}{4}w^2\right)v_x\right]\partial_x^{-1}v_x + \tfrac{1}{4}v_x\partial_x^{-1}\left[uw_x - \left(v - \tfrac{1}{4}w^2\right)v_x\right] \\
\left(\pi_0^{\mathrm{red}}\right)_{23} &= u\left(v - \tfrac{1}{4}w^2\right)\partial_x - \tfrac{1}{2}\left[u - w\left(v - \tfrac{1}{4}w^2\right)\right]v_x - \tfrac{1}{4}uww_x \\
&\quad + \tfrac{1}{4}v_x\partial_x^{-1}\left[2u_x - (vw)_x + \tfrac{1}{4}(w^3)_x\right] + \tfrac{1}{4}\left[uw_x - \left(v - \tfrac{1}{4}w^2\right)v_x\right]\partial_x^{-1}w_x
\end{aligned}$$

$$\left(\pi_0^{\mathrm{red}}\right)_{33} = \tfrac{1}{4}[(vw^2 - 2uw)\partial_x + \partial_x(vw^2 - 2uw)] - \tfrac{1}{16}(w^4\partial_x + \partial_x w^4) + \tfrac{1}{4}\left(2u - vw + \tfrac{1}{4}w^3\right)_x \partial_x^{-1} w_x + \tfrac{1}{4} w_x \partial_x^{-1}\left(2u - vw + \tfrac{1}{4}w^3\right)_x .$$

Note that both Poisson structures are nonlocal. Then,

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{t_2} = 2\begin{pmatrix} u^2 v_x \\ u u_x w + u^2 w_x \\ 2u u_x \end{pmatrix} = \pi_{-1}^{\mathrm{red}}\,\mathrm{d}H_1 = \pi_0^{\mathrm{red}}\,\mathrm{d}H_0 \tag{4.60}$$

where

$$H_1 = \int_\Omega (uw^2 + v^2 w + 2uv)\,\mathrm{d}x \qquad H_0 = \int_\Omega (u + vw)\,\mathrm{d}x. \tag{4.61}$$

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